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γ - P_S -Functions in Topological Spaces

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Abstract

This paper introduces some new classes of functions called γ - P_S -continuous, β - P_S -open and β - P_S -closed using γ - P_S -open set and γ - P_S -closed set. In addition, some properties and characterizations of these functions are given. The result shows that γ - P_S -continuous function and γ -continuous function are independent.

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1 Introduction

Kasahara [5] defined the concept of α -closed graphs of an operation γ on τ . Later, Ogata [12] renamed the operation α as γ operation on τ . He defined and investigated the concept of operation-open sets, that is, γ -open sets. Further study by Krishnan and Balachandran ([8], [9]) defined two types of sets called γ -preopen and γ -semiopen sets. Recently, Asaad, Ahmad and Omar [1] defined the notion of γ -regular-open sets which lies strictly between the classes of γ -open set and γ -clopen set. They also introduced a new class of sets called γ - P_S -open sets, and they also defined γ - P_S -operations and their properties [2].

They proved that the union of any class of γ - P_S -open sets in a space X is also a γ - P_S -open, but the intersection of any two γ - P_S -open sets may not be a γ - P_S -open.

In this paper, we define the concept of γ - P_S -continuous function and then establish its properties. The result reveals that γ - P_S -continuous function and γ -continuous function are independent. Furthermore two other classes of functions called β - P_S -open and β - P_S -closed are defined. Some properties and theorems for these two functions are also presented.

Throughout this paper, the pairs (X, τ) and (Y, σ) (or simply X and Y) represent denote topological spaces with no separation axioms assumed unless explicitly stated. Let A be any subset of X , $Int(A)$ and $Cl(A)$ denotes the interior of A and the closure of A , respectively.

2 Preliminaries

A subset A of X is said to be preopen if $A \subseteq Int(Cl(A))$ [11] and semiopen if $A \subseteq Cl(Int(A))$ [10]. The complement of a semiopen set is said to be semiclosed. A preopen subset A of a topological space (X, τ) is said to be P_S -open if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$ [6]. An operation γ on the topology τ on X is a mapping $\gamma: \tau \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [12]. A nonempty set A of X with an operation γ on τ is said to be γ -open [12] if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ -open set is called a γ -closed. The τ_γ -closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ -closed sets containing A and it is denoted by $\tau_\gamma Cl(A)$ [12], and the τ_γ -interior of a subset A of X with an operation γ on τ is defined as the union of all γ -open sets containing A [9].

A topological space (X, τ) with an operation γ on τ is said to be γ -regular if for each $x \in X$ and for each open neighborhood V of x , there exists an open neighborhood U of x such that $\gamma(U) \subseteq V$ [5]. A topological space (X, τ) with an operation γ on τ is said to be γ -locally indiscrete if every γ -open subset of X is γ -closed, or every γ -closed subset of X is γ -open [1]. A topological space (X, τ) with an operation γ on τ is said to be γ -semi- T_1 if for each pair of distinct points x, y in X , there exist two γ -semiopen sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ [9]. Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called γ -continuous if $f^{-1}(V)$ is γ -open set in X , for every open set V in Y [3].

Now we recall some definitions and results which will be used in the sequel.

Remark 2.1 For any subset A of a topological space (X, τ) . Then:

1. A is γ -open if and only if $\tau_\gamma\text{-Int}(A) = A$ [7].
2. A is γ -closed if and only if $\tau_\gamma\text{-Cl}(A) = A$ [12].

Definition 2.2 Let (X, τ) be a topological space and γ be an operation on τ . A subset A of X is said to be:

1. γ -regular-open if $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A))$ [1].
2. γ -preopen if $A \subseteq \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A))$ [8].
3. γ -semiopen if $A \subseteq \tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A))$ [9].
4. γ - β -open if $A \subseteq \tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A)))$ [3].

Definition 2.3 The complement of γ -regular-open, γ -preopen and γ -semiopen set is said to be γ -regular-closed [3], γ -preclosed [8] and γ -semiclosed [9], respectively.

Definition 2.4 [2] A γ -preopen subset A of a topological space (X, τ) is called γ - P_S -open if for each $x \in A$, there exists a γ -semiclosed set F such that $x \in F \subseteq A$. The complement of a γ - P_S -open set is called a γ - P_S -closed.

The family of all γ - P_S -open and γ -preopen subsets of a topological space (X, τ) are denoted by $\tau_\gamma\text{-}P_SO(X)$ and $\tau_\gamma\text{-}PO(X)$, respectively.

Lemma 2.5 [2] Let A be a subset of a topological space (X, τ) and γ be an operation on τ . Then A is γ - P_S -open if and only if A is γ -preopen set and A is a union of γ -semiclosed sets.

Definition 2.6 [2] Let A be any subset of a topological space (X, τ) and γ be an operation on τ . Then

1. the $\tau_\gamma\text{-}P_S$ -interior of A is defined as the union of all γ - P_S -open sets of X contained in A and it is denoted by $\tau_\gamma\text{-}P_S\text{Int}(A)$.
2. the $\tau_\gamma\text{-}P_S$ -closure of A is defined as the intersection of all γ - P_S -closed sets of X containing A and it is denoted by $\tau_\gamma\text{-}P_S\text{Cl}(A)$.

Definition 2.7 [2] A subset N of a topological space (X, τ) is called a γ - P_S -neighbourhood of a point $x \in X$, if there exists a γ - P_S -open set U in X such that $x \in U \subseteq N$.

Definition 2.8 [2] Let (X, τ) be a topological space and γ be an operation on τ . Let A be any subset of X . Then

1. the γ - P_S -derived set of A is defined as $\{x : \text{for every } \gamma\text{-open set } U \text{ containing } x, U \cap A \setminus \{x\} \neq \emptyset\}$ and it is denoted by $\tau_\gamma\text{-}P_S D(A)$.
2. the γ - P_S -boundary of A is defined as $\tau_\gamma\text{-}P_S Cl(A) \setminus \tau_\gamma\text{-}P_S Int(A)$ and it is denoted by $\tau_\gamma\text{-}P_S Bd(A)$.

Theorem 2.9 [2] Let A be a subset of a topological space (X, τ) and γ be an operation on τ . Then $x \in \tau_\gamma\text{-}P_S Cl(A)$ if and only if $A \cap U \neq \emptyset$ for every γ - P_S -open set U of X containing x .

Theorem 2.10 [2] Let (X, τ) be a topological space and γ be an operation on τ . For any subset A of a space X . The following statements are true.

1. A is γ - P_S -open set if and only if $\tau_\gamma\text{-}P_S Int(A) = A$.
2. A is γ - P_S -closed set if and only if $\tau_\gamma\text{-}P_S Cl(A) = A$.
3. $\tau_\gamma\text{-}P_S Cl(X \setminus A) = X \setminus \tau_\gamma\text{-}P_S Int(A)$ and $\tau_\gamma\text{-}P_S Int(X \setminus A) = X \setminus \tau_\gamma\text{-}P_S Cl(A)$.
4. $\tau_\gamma\text{-}P_S D(A) \subseteq \tau_\gamma\text{-}P_S Cl(A)$.
5. $\tau_\gamma\text{-}P_S Cl(A) = \tau_\gamma\text{-}P_S Int(A) \cup \tau_\gamma\text{-}P_S Bd(A)$.

Remark 2.11 If a topological space (X, τ) is γ -regular, then $\tau_\gamma = \tau$ [12] and hence $\tau_\gamma\text{-}Int(A) = Int(A)$ [7].

Theorem 2.12 [2] Let (X, τ) be a topological space and γ be an operation on τ . Then:

1. If X is γ -regular, then the concept of γ - P_S -open set and P_S -open set are equivalent.
2. If X is γ -semi- T_1 , then the concept of γ - P_S -open set and γ -preopen set are equivalent.
3. If X is γ -locally indiscrete, then the concept of γ - P_S -open set and γ -open set are equivalent.

Definition 2.13 [6] A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be P_S -continuous if the inverse image of each open set in Y is P_S -open in X .

Definition 2.14 [4] Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called γ -precontinuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ -preopen set U of X containing x such that $f(U) \subseteq V$. If f is γ -precontinuous at each point x of X , then f is said to be γ -precontinuous.

Theorem 2.15 [4] Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ -precontinuous if for every open set V in Y , $f^{-1}(V)$ is γ -preopen set in X .

Definition 2.16 [4] Let β be an operation on a topological space (Y, σ) . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called β -preopen and β -preclosed if for every open and closed set V of X , $f(V)$ is β -preopen and β -preclosed set in Y , respectively.

3 γ - P_S -Continuous Functions

In this section, we introduce a new class of functions called γ - P_S -continuous using γ - P_S -open set. Moreover, we give some characterizations and theorems of this function. The result shows that γ - P_S -continuous and γ -continuous functions are independent.

Definition 3.1 Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called γ - P_S -continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ - P_S -open set U of X containing x such that $f(U) \subseteq V$. If f is γ - P_S -continuous at every point x in X , then f is said to be γ - P_S -continuous.

Theorem 3.2 For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ and γ be an operation on τ , the following statements are equivalent:

1. f is γ - P_S -continuous.
2. $f^{-1}(V)$ is γ - P_S -open set in X , for every open set V in Y .
3. $f^{-1}(F)$ is γ - P_S -closed set in X , for every closed set F in Y .
4. $f(\tau_\gamma\text{-}P_S\text{Cl}(A)) \subseteq \text{Cl}(f(A))$, for every subset A of X .
5. $\tau_\gamma\text{-}P_S\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B))$, for every subset B of Y .
6. $f^{-1}(\text{Int}(B)) \subseteq \tau_\gamma\text{-}P_S\text{Int}(f^{-1}(B))$, for every subset B of Y .
7. $\text{Int}(f(A)) \subseteq f(\tau_\gamma\text{-}P_S\text{Int}(A))$, for every subset A of X .

Proof: (1) \Rightarrow (2) Let V be any open set in Y . We have to show that $f^{-1}(V)$ is γ - P_S -open set in X . Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (1), there exists a γ - P_S -open set U of X containing x such that $f(U) \subseteq V$. Which implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is γ - P_S -open set in X .

(2) \Rightarrow (3). Let F be any closed set of Y . Then $Y \setminus F$ is an open set of Y . By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is γ - P_S -open set in X and hence $f^{-1}(F)$ is γ - P_S -closed set in X .

(3) \Rightarrow (4). Let A be any subset of X . Then $f(A) \subseteq Cl(f(A))$ and hence $A \subseteq f^{-1}(Cl(f(A)))$. Since $Cl(f(A))$ is closed set in Y . Then by (3), we have $f^{-1}(Cl(f(A)))$ is γ - P_S -closed set in X . Therefore, τ_γ - P_S $Cl(A) \subseteq f^{-1}(Cl(f(A)))$. Hence $f(\tau_\gamma$ - P_S $Cl(A)) \subseteq Cl(f(A))$.

(4) \Rightarrow (5). Let B be any subset of Y . Then $f^{-1}(B)$ is a subset of X . By (4), we have $f(\tau_\gamma$ - P_S $Cl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) = Cl(B)$. Hence τ_γ - P_S $Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(5) \Leftrightarrow (6). Let B be any subset of Y . Then apply (5) to $Y \setminus B$ we obtain τ_γ - P_S $Cl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B)) \Leftrightarrow \tau_\gamma$ - P_S $Cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B)) \Leftrightarrow X \setminus \tau_\gamma$ - P_S $Int(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq \tau_\gamma$ - P_S $Int(f^{-1}(B))$. Therefore, $f^{-1}(Int(B)) \subseteq \tau_\gamma$ - P_S $Int(f^{-1}(B))$.

(6) \Rightarrow (7). Let A be any subset of X . Then $f(A)$ is a subset of Y . By (6), we have $f^{-1}(Int(f(A))) \subseteq \tau_\gamma$ - P_S $Int(f^{-1}(f(A))) = \tau_\gamma$ - P_S $Int(A)$. Therefore, $Int(f(A)) \subseteq f(\tau_\gamma$ - P_S $Int(A))$.

(7) \Rightarrow (1). Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X . By (7), we have $Int(f(f^{-1}(V))) \subseteq f(\tau_\gamma$ - P_S $Int(f^{-1}(V)))$. Then $Int(V) \subseteq f(\tau_\gamma$ - P_S $Int(f^{-1}(V)))$. Since V is an open set. Then $V \subseteq f(\tau_\gamma$ - P_S $Int(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq \tau_\gamma$ - P_S $Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ - P_S -open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is γ - P_S -continuous function.

Theorem 3.3 Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be any function and γ be an operation on τ . Then f is γ - P_S -continuous if and only if τ_γ - P_S $Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$, for each subset B of Y .

Proof: Let B be any subset of Y and f be a γ - P_S -continuous function. Then by using Theorem 3.2 (2) and (5), we have $f^{-1}(Bd(B)) = f^{-1}(Cl(B) \setminus Int(B)) = f^{-1}(Cl(B)) \setminus f^{-1}(Int(B)) \supseteq \tau_\gamma$ - P_S $Cl(f^{-1}(B)) \setminus f^{-1}(Int(B)) = \tau_\gamma$ - P_S $Cl(f^{-1}(B)) \setminus \tau_\gamma$ - P_S $Int(f^{-1}(Int(B))) \supseteq \tau_\gamma$ - P_S $Cl(f^{-1}(B)) \setminus \tau_\gamma$ - P_S $Int(f^{-1}(B)) = \tau_\gamma$ - P_S $Bd(f^{-1}(B))$. Hence τ_γ - P_S $Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$.

Conversely, let G be any open set in Y . Then $Y \setminus G$ is closed in Y . So by hypothesis, we have τ_γ - P_S $Bd(f^{-1}(Y \setminus G)) \subseteq f^{-1}(Bd(Y \setminus G)) \subseteq f^{-1}(Cl(Y \setminus G)) = f^{-1}(Y \setminus G)$. By Theorem 2.10 (5), τ_γ - P_S $Cl(f^{-1}(Y \setminus G)) = \tau_\gamma$ - P_S $Int(f^{-1}(Y \setminus G)) \cup \tau_\gamma$ - P_S $Bd(f^{-1}(Y \setminus G)) \subseteq f^{-1}(Y \setminus G)$. Then $f^{-1}(Y \setminus G)$ is γ - P_S -closed set in X and hence $f^{-1}(G)$ is γ - P_S -open set in X . By Theorem 3.2, f is γ - P_S -continuous function.

Theorem 3.4 *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be any function and γ be an operation on τ . Then f is γ - P_S -continuous if and only if $f(\tau_\gamma\text{-}P_S D(A)) \subseteq Cl(f(A))$, for each subset A of X .*

Proof: Let f be a γ - P_S -continuous function and A be any subset of X . Then by Theorem 3.2 (4), we have $f(\tau_\gamma\text{-}P_S Cl(A)) \subseteq Cl(f(A))$ and by Theorem 2.10 (4), $f(\tau_\gamma\text{-}P_S D(A)) \subseteq f(\tau_\gamma\text{-}P_S Cl(A))$ which implies that $f(\tau_\gamma\text{-}P_S D(A)) \subseteq Cl(f(A))$.

Conversely, let F be any closed set in Y . Then $f^{-1}(F)$ is subset of X . By hypothesis, we have $f(\tau_\gamma\text{-}P_S D(f^{-1}(F))) \subseteq Cl(f(f^{-1}(F))) = Cl(F) = F$ and hence $\tau_\gamma\text{-}P_S D(f^{-1}(F)) \subseteq f^{-1}(F)$. Then $f^{-1}(F)$ is γ - P_S -closed set in X . Therefore, by Theorem 3.2, f is γ - P_S -continuous function.

Theorem 3.5 *Let γ be an operation on (X, τ) . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - P_S -continuous if and only if for every $x \in X$ and for neighbourhood O of Y such that $f(x) \in O$, there exists a γ - P_S -neighbourhood P of X such that $x \in P$ and $f(P) \subseteq O$.*

Proof: It is clear and hence it is omitted.

Theorem 3.6 *Let γ be an operation on (X, τ) . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - P_S -continuous if and only if the inverse image of every neighbourhood of $f(x)$ is γ - P_S -neighbourhood of $x \in X$.*

Proof: The proof follows from Theorem 3.5.

Theorem 3.7 *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjection function and γ be an operation on τ , then the following statements are equivalent:*

1. f is γ - P_S -continuous.
2. $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X , for every subset B in Y .
3. $Cl(Int(f^{-1}(B))) \subseteq f^{-1}(Cl(B))$ and $f^{-1}(Cl(B)) = \cup_{i \in I} G_i$ where G_i is γ -semiopen set in X , for every subset B in Y .
4. $f(Cl(Int(A))) \subseteq Cl(f(A))$ and $f^{-1}(Cl(f(A))) = \cup_{i \in I} G_i$ where G_i is γ -semiopen set in X , for every subset A in X .
5. $Int(f(A)) \subseteq f(Int(Cl(A)))$ and $f^{-1}(Int(f(A))) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X , for every subset A in X .

Proof: It is enough to proof (1) \Rightarrow (2) and (5) \Rightarrow (1) since the others are obvious.

(1) \Rightarrow (2). Let B be any subset in Y . Then $Int(B)$ is open set in Y . Since f is γ - P_S -continuous, then by Theorem 3.2, $f^{-1}(Int(B))$ is γ - P_S -open set in X . By Lemma 2.5, we obtain $f^{-1}(Int(B))$ is γ -preopen set in X and $f^{-1}(Int(B)) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X , for every subset B in Y . Therefore, $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(Int(B))))$ and $f^{-1}(Int(B)) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X . Thus $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X .

(5) \Rightarrow (1). Let V be any open set in Y . Then $f^{-1}(V)$ is a subset of X . By (5), we get $Int(f(f^{-1}(V))) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(f(f^{-1}(V)))) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X . Hence $Int(V) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(V)) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X . This implies that $V \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(V) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X and hence $f^{-1}(V) \subseteq Int(Cl(f^{-1}(V)))$ and $f^{-1}(V) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X . So $f^{-1}(V)$ is γ -preopen set in X and $f^{-1}(V) = \cup_{i \in I} F_i$ where F_i is γ -semiclosed set in X . Therefore, by Lemma 2.5, $f^{-1}(V)$ is γ - P_S -open set in X and hence by Theorem 3.2, f is γ - P_S -continuous.

Remark 3.8 Every γ - P_S -continuous function is γ -precontinuous, but the converse is not true as it is shown in the following example.

Example 3.9 Let $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$. Define an operation $\gamma: \tau \rightarrow P(X)$ as follows: for every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ Cl(A) & \text{if } a \notin A \end{cases}$$

Then $\tau_\gamma = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$, τ_γ - $PO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and τ_γ - $P_S O(X) = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$. Let $f: (X, \tau) \rightarrow (X, \sigma)$ be a function defined as follows: $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is γ -precontinuous, but it is not γ - P_S -continuous since $\{a\} \in \sigma$, but $f^{-1}(\{a\}) = \{b\} \notin \tau_\gamma$ - $P_S O(X)$.

Theorem 3.10 Let (X, τ) be γ -semi- T_1 space and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - P_S -continuous if and only if f is γ -precontinuous.

Proof: This is an immediate consequence of Theorem 2.12 (2).

Theorem 3.11 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation γ on τ is γ - P_S -continuous if and only if f is γ -precontinuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ -semiclosed set F in X containing x such that $f(F) \subseteq V$.

Proof: Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since f is γ - P_S -continuous, there exists a γ - P_S -open set U of X containing x such that $f(U) \subseteq V$. Since U is γ - P_S -open set. Then for each $x \in U$, there exists a γ -semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we get $f(F) \subseteq V$. And also since f is γ - P_S -continuous. Then f is γ -precontinuous.

Conversely, let V be any open set of Y . We have to show that $f^{-1}(V)$ is γ - P_S -open set in X . Since f is γ -precontinuous, then by Theorem 2.15, $f^{-1}(V)$ is γ -preopen set in X . Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a γ -semiclosed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 2.4, $f^{-1}(V)$ is γ - P_S -open set in X . Hence by Theorem 3.2, f is γ - P_S -continuous.

Theorem 3.12 *If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation γ on τ is γ - P_S -continuous, then for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ -semiclosed set F in X such that $x \in F$ and $f(F) \subseteq V$.*

Proof: Suppose f be a γ - P_S -continuous function and let V be any open set of Y such that $f(x) \in V$, for each $x \in X$. Then there exists a γ - P_S -open set U of X such that $x \in U$ and $f(U) \subseteq V$. Since U is γ - P_S -open set. Then for each $x \in U$, there exists a γ -semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. This completes the proof.

Theorem 3.13 *Let (X, τ) be γ -regular space and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - P_S -continuous if and only if f is P_S -continuous.*

Proof: This is an immediate consequence of Theorem 2.12 (1).

The following example shows that the relation between γ - P_S -continuous function and γ -continuous function are independent in general.

Example 3.14 *Let (X, τ) be a topological space and γ be an operation on τ as in Example 3.9. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{2\}, \{2, 3\}\}$ be a topology on Y . Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Then f is γ -continuous, but it is not γ - P_S -continuous since $\{2, 3\} \in \sigma$, but $f^{-1}(\{2, 3\}) = \{a, b\} \notin \tau_{\gamma}$.*

Example 3.15 *Let (X, τ) be a topological space and γ be an operation on τ as in Example 3.9. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{1, 3\}\}$ be a topology on Y . Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Then f is γ - P_S -continuous, but it is not γ -continuous since $\{1, 3\} \in \sigma$, but $f^{-1}(\{1, 3\}) = \{a, c\} \notin \tau_{\gamma}$.*

Theorem 3.16 *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and let (X, τ) be γ -locally indiscrete topological spaces and γ be an operation on τ . Then f is γ - P_S -continuous if and only if f is γ -continuous.*

Proof: This is an immediate consequence of Theorem 2.12 (3).

Theorem 3.17 *For any operation γ on τ and $f: (X, \tau) \rightarrow (Y, \sigma)$ be any function, then: $X \setminus \tau_\gamma\text{-}P_S C(f) = \cup\{\tau_\gamma\text{-}P_S B_d(f^{-1}(V)) : V \text{ is open in } (Y, \sigma) \text{ such that } f(x) \in V \text{ for each } x \in X\}$, where $\tau_\gamma\text{-}P_S C(f)$ denotes the set of points at which f is γ - P_S -continuous function.*

Proof: Let $x \in \tau_\gamma\text{-}P_S C(f)$. Then there exists open set V in (Y, σ) containing $f(x)$ such that $f(U) \not\subseteq V$ for every γ - P_S -open set U of (X, τ) containing x . Hence $U \cap X \setminus f^{-1}(V) \neq \emptyset$ for every γ - P_S -open set U of (X, τ) containing x . Therefore, by Theorem 2.9, $x \in \tau_\gamma\text{-}P_S Cl(X \setminus f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \tau_\gamma\text{-}P_S Cl(X \setminus f^{-1}(V)) \subseteq \tau_\gamma\text{-}P_S Cl(f^{-1}(V)) \cap \tau_\gamma\text{-}P_S Cl(X \setminus f^{-1}(V)) = \tau_\gamma\text{-}P_S B_d(f^{-1}(V))$. Then $X \setminus \tau_\gamma\text{-}P_S C(f) \subseteq \cup\{\tau_\gamma\text{-}P_S B_d(f^{-1}(V)) : V \text{ is open in } (Y, \sigma) \text{ such that } f(x) \in V \text{ for each } x \in X\}$.

Conversely, let $x \notin X \setminus \tau_\gamma\text{-}P_S C(f)$. Then for each open set V in (Y, σ) containing $f(x)$, $f^{-1}(V)$ is γ - P_S -open set of (X, τ) containing x . Hence $x \in \tau_\gamma\text{-}P_S Int(f^{-1}(V))$ and hence $x \notin \tau_\gamma\text{-}P_S B_d(f^{-1}(V))$ for every open set V in (Y, σ) containing $f(x)$. Therefore, $X \setminus \tau_\gamma\text{-}P_S C(f) \supseteq \cup\{\tau_\gamma\text{-}P_S B_d(f^{-1}(V)) : V \text{ is open in } (Y, \sigma) \text{ such that } f(x) \in V \text{ for each } x \in X\}$.

Theorem 3.18 *Let γ be an operation on the topological space (X, τ) . If the functions $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - P_S -continuous and $g: (Y, \sigma) \rightarrow (Z, \rho)$ is continuous. Then the composition function $g \circ f: (X, \tau) \rightarrow (Z, \rho)$ is γ - P_S -continuous.*

Proof: Clear.

Proposition 3.19 *Let γ be an operation on the topological space (X, τ) . If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a function, $g: (Y, \sigma) \rightarrow (Z, \rho)$ is open and injective, and $g \circ f: (X, \tau) \rightarrow (Z, \rho)$ is γ - P_S -continuous. Then f is γ - P_S -continuous.*

Proof: Let V be an open subset of Y . Since g is open, $g(V)$ is open subset of Z . Since $g \circ f$ is γ - P_S -continuous and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V))) = (g \circ f)^{-1}(g(V))$ is γ - P_S -open in X , which proves that f is γ - P_S -continuous.

Definition 3.20 *A subset A of a topological space (X, τ) with an operation γ on τ is called γ - γ - P_S -open and τ - γ - P_S -open if $\tau_\gamma\text{-}Int(A) = \tau_\gamma\text{-}P_S Int(A)$ and $Int(A) = \tau_\gamma\text{-}P_S Int(A)$, respectively.*

Lemma 3.21 *Let A be any subset of a topological space (X, τ) and γ be an operation on τ . Then the following statements are equivalent:*

1. A is γ - γ - P_S -open and γ - P_S -open.
2. A is γ - γ - P_S -open and γ -open.
3. A is γ - P_S -open and γ -open.

Proof: Follows from Definition 3.20, Remark 2.1 (1) and Theorem 2.10 (1).

Lemma 3.22 *Let A be any subset of a topological space (X, τ) and γ be an operation on τ . Then the following statements are equivalent:*

1. A is τ - γ - P_S -open and γ - P_S -open.
2. A is τ - γ - P_S -open and open.
3. A is γ - P_S -open and open.

Proof: Follows from Definition 3.20 and Theorem 2.10 (1).

Proposition 3.23 *In a γ -regular space (X, τ) , then the concept of γ - γ - P_S -open set and τ - γ - P_S -open set are equivalent.*

Proof: The proof follows from Definition 3.20 and Remark 2.11.

Definition 3.24 *Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called τ - γ - P_S -continuous and γ - γ - P_S -continuous if for each open set V of Y , $f^{-1}(V)$ is τ - γ - P_S -open and γ - γ - P_S -open sets in X , respectively.*

Theorem 3.25 *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ and γ be an operation on τ , the following statements are equivalent:*

1. f is γ - γ - P_S -continuous and γ - P_S -continuous.
2. f is γ - γ - P_S -continuous and γ -continuous.
3. f is γ - P_S -continuous and γ -continuous.

Proof: The proof follows from Lemma 3.21.

Theorem 3.26 *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ and γ be an operation on τ , the following statements are equivalent:*

1. f is τ - γ - P_S -continuous and γ - P_S -continuous.

2. f is τ - γ - P_S -continuous and continuous.
3. f is γ - P_S -continuous and continuous.

Proof: Follows from Lemma 3.22.

Theorem 3.27 *Let (X, τ) be γ -regular space and γ be an operation on τ . Then a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is γ - γ - P_S -continuous if and only if f is τ - γ - P_S -continuous.*

Proof: This is an immediate consequence of Proposition 3.23.

4 β - P_S -Open and β - P_S -Closed Functions

Definition 4.1 *Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called β - P_S -open if for every open set V of X , $f(V)$ is β - P_S -open set in Y .*

Definition 4.2 *Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ called β - P_S -closed if for every closed set F of X , $f(F)$ is β - P_S -closed set in Y .*

Theorem 4.3 *Let β be an operation on (Y, σ) . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - P_S -open if and only if for every $x \in X$ and for every neighbourhood N of x , there exists a β - P_S -neighbourhood M of $f(x)$ such that $f(x) \in M$ and $M \subseteq f(N)$.*

Proof: Obvious.

Theorem 4.4 *The following statements are equivalent for a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation β on σ :*

1. f is β - P_S -open.
2. $f(Int(A)) \subseteq \sigma_\beta\text{-}P_S Int(f(A))$, for every $A \subseteq X$.
3. $Int(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_S Int(B))$, for every $B \subseteq Y$.
4. $f^{-1}(\sigma_\beta\text{-}P_S Cl(B)) \subseteq Cl(f^{-1}(B))$, for every $B \subseteq Y$.
5. $\sigma_\beta\text{-}P_S Cl(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.
6. $\sigma_\beta\text{-}P_S D(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.

Proof: The proof is similar to Theorem 3.2.

Remark 4.5 Every β - P_S -open and β - P_S -closed function is β -preopen and β -preclosed respectively, but the converse is not true as it is shown in the following example.

Example 4.6 Let $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\phi, X, \{b\}, \{a, c\}\}$. Define an operation β on σ by $\beta(A) = A$ for all $A \in \sigma$. Define $f: (X, \tau) \rightarrow (X, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is both β -preopen and β -preclosed, but f is not β - P_S -open and β - P_S -closed function since $\{b, c\} \in \tau$ and $\{a\}$ is closed set in (X, τ) , but $f(\{b, c\}) = \{a, c\}$ is not β - P_S -open set in (X, σ) and $f(\{a\}) = \{a\}$ is not β - P_S -closed set in (X, σ) , respectively.

Theorem 4.7 Let (Y, σ) be β -semi- T_1 space and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - P_S -open if and only if f is β -preopen.

Proof: Follows from Theorem 2.12 (2).

Definition 4.8 Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called β -open if for every open set V of X , $f(V)$ is β -open set in Y .

Theorem 4.9 Let (Y, σ) be β -locally indiscrete topological spaces and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - P_S -open if and only if f is β -open.

Proof: Follows from Theorem 2.12 (3).

Theorem 4.10 Let (Y, σ) be a topological space and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - P_S -closed if and only if for each subset S of Y and each open set O in X containing $f^{-1}(S)$, there exists a β - P_S -open set R in Y containing S such that $f^{-1}(R) \subseteq O$.

Proof: Suppose that f is β - P_S -closed function and let O be an open set in X containing $f^{-1}(S)$, where S is any subset in Y . Then $f(X \setminus O)$ is β - P_S -open set in Y . If we put $R = Y \setminus f(X \setminus O)$. Then R is β - P_S -closed set in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Conversely, let F be closed set in X . Let $S = Y \setminus f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X \setminus F$ and $X \setminus F$ is open set in X . By hypothesis, there exists a β - P_S -open set R in Y such that $S = Y \setminus f(F) \subseteq R$ and $f^{-1}(R) \subseteq X \setminus F$. For $f^{-1}(R) \subseteq X \setminus F$ implies $R \subseteq f(X \setminus F) \subseteq Y \setminus f(F)$. Hence $R = Y \setminus f(F)$. Since R is β - P_S -open set in Y . Then $f(F)$ is β - P_S -closed set in Y . Therefore, f is β - P_S -closed function.

Definition 4.11 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be γ - P_S -homeomorphism, if f is bijective, γ - P_S -continuous and f^{-1} is γ - P_S -continuous.

Theorem 4.12 *The following statements are equivalent for a bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation β on σ .*

1. f is β - P_S -closed.
2. f is β - P_S -open.
3. f^{-1} is β - P_S -continuous.

Proof: It is clear.

Proposition 4.13 *Let α be an operation on the topological space (Z, ρ) . If the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is open (resp., closed) and $g: (Y, \sigma) \rightarrow (Z, \rho)$ is α - P_S -open (resp., α - P_S -closed). Then the composition function $g \circ f: (X, \tau) \rightarrow (Z, \rho)$ is α - P_S -open (resp., α - P_S -closed).*

Proof: Obvious.

Proposition 4.14 *Let β be an operation on the topological space (Y, σ) . If $g: (Y, \sigma) \rightarrow (Z, \rho)$ is a function, $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - P_S -open and surjective, and $g \circ f: (X, \tau) \rightarrow (Z, \rho)$ is continuous. Then g is γ - P_S -continuous.*

Proof: Similar to Proposition 3.19.

Proposition 4.15 *Let β be an operation on the topological space (Y, σ) . If $g: (Y, \sigma) \rightarrow (Z, \rho)$ is a function, $f: (X, \tau) \rightarrow (Y, \sigma)$ is continuous and surjective, and $g \circ f: (X, \tau) \rightarrow (Z, \rho)$ is β - P_S -open. Then g is β - P_S -open.*

Proof: Similar to Proposition 3.19.

Definition 4.16 *Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ . A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called σ - β - P_S -open and β - β - P_S -open if for every open set V of X , $f(V)$ is σ - β - P_S -open and β - β - P_S -open sets in Y , respectively.*

Theorem 4.17 *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ and β be an operation on σ , the following statements are equivalent:*

1. f is β - β - P_S -open (resp., β - β - P_S -closed) and β - P_S -open (resp., β - P_S -closed).
2. f is β - β - P_S -open (resp., β - β - P_S -closed) and β -open (resp., β -closed).
3. f is β - P_S -open (resp., β - P_S -closed) and β -open (resp., β -closed).

Proof: Follows from Lemma 3.21.

Theorem 4.18 For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ and β be an operation on σ , the following statements are equivalent:

1. f is σ - β - P_S -open (resp., σ - β - P_S -closed) and β - P_S -open (resp., β - P_S -closed).
2. f is σ - β - P_S -open (resp., σ - β - P_S -closed) and open (resp., closed).
3. f is β - P_S -open (resp., β - P_S -closed) and open (resp., closed).

Proof: Follows from Lemma 3.22.

Theorem 4.19 Let (Y, σ) be β -regular space and β be an operation on σ . Then a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β - β - P_S -open (resp., β - β - P_S -closed) if and only if f is σ - β - P_S -open (resp., σ - β - P_S -closed).

Proof: This is an immediate consequence of Proposition 3.23.

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